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CHAPTER 8: THE DISJOINT SET ADT

In this chapter, we describe an efficient data structure to solve the equivalence problem. The data structure is simple to implement. Each routine requires only a few lines of code, and a simple array can be used. The implementation is also extremely fast, requiring constant average time per operation. This data structure is also very interesting from a theoretical point of view, because its analysis is extremely difficult; the functional form of the worst case is unlike any we have yet seen. For the disjoint set ADT, we will

Show how it can be implemented with minimal coding effort.

Greatly increase its speed, using just two simple observations.

Analyze the running time of a fast implementation.

See a simple application.

8.1. Equivalence Relations

A *relation R* is defined on a set *S* if for every pair of elements (*a, b*), *a, b* *S, a R b* is either true or false. If *a R b* is true, then we say that *a* is related to *b*.

An *equivalence relation* is a relation *R* that satisfies three properties:

1. (*Reflexive*) *a R a*, for all *a* S.

2. (*Symmetric*) *a R b* if and only if *b R a*.

3. (*Transitive*) *a R b* and *b R c* implies that *a R c*.

We'll consider several examples.

The relationship is not an equivalence relationship. Although it is reflexive, since *a* *a*, and transitive, since *a* b and *b* *c* implies *a* *c*, it is not symmetric, since *a* *b* does not imply *b* *a*.

Electrical connectivity, where all connections are by metal wires, is an equivalence relation. The relation is clearly reflexive, as any component is connected to itself. If a is electrically connected to b, then b must be electrically connected to a, so the relation is symmetric. Finally, if a is connected to b and b is connected to c, then a is connected to c. Thus electrical connectivity is an equivalence relation.

Two cities are related if they are in the same country. It is easily verified that this is an equivalence relation. Suppose town a is related to b if it is possible to travel from a to b by taking roads. This relation is an equivalence relation if all the roads are two-way.

8.2. The Dynamic Equivalence Problem

Given an equivalence relation $\sim$, the natural problem is to decide, for any a and b, if $a \sim b$. If the relation is stored as a two-dimensional array of booleans, then, of course, this can be done in constant time. The problem is that the relation is usually not explicitly, but rather implicitly, defined.

As an example, suppose the equivalence relation is defined over the five-element set $\{a_1, a_2, a_3, a_4, a_5\}$. Then there are 25 pairs of elements, each of which is either related or not. However, the information $a_1 \sim a_2$, $a_3 \sim a_4$, $a_5 \sim a_1$, $a_4 \sim a_2$ implies that all pairs are related. We would like to be able to infer this quickly.

The *equivalence class* of an element a S is the subset of S that contains all the elements that are related to a. Notice that the equivalence classes form a partition of S: Every member of S appears in exactly one equivalence class. To decide if $a \sim b$, we need only to check whether a and b are in the same equivalence class. This provides our strategy to solve the equivalence problem.

The input is initially a collection of n sets, each with one element. This initial representation is that all relations (except reflexive relations) are false. Each set has a different element, so that S_i S_j = ; this makes the sets *disjoint*.

There are two permissible operations. The first is *find*, which returns the name of the set (that is, the equivalence class) containing a given element. The second operation adds relations. If we want to add the relation $a \sim b$, then we first see if a and b are already related. This is done by performing *find*s on both a and b and checking whether they are in the same equivalence class. If they are not, then we apply *union*. This operation merges the two equivalence classes containing a and b into a new equivalence class. From a set point of view, the result of is to create a new set $S_k = S_i$ S_j, destroying the originals and preserving the disjointness of all the sets. The algorithm to do this is frequently known as the disjoint set *union/find algorithm* for this reason.

This algorithm is *dynamic* because, during the course of the algorithm, the sets can change via the *union* operation. The algorithm must also operate *on-line*: When a *find* is performed, it must give an answer before continuing. Another possibility would be an *off-line* algorithm. Such an algorithm would be allowed to see the entire sequence of *unions* and *finds*. The answer it provides for each *find* must still be consistent with all the *unions* that were performed up until the *find*, but the algorithm can give all its answers after it has seen *all* the questions. The difference is similar to taking a written exam (which is generally off-line--you only have to give the answers before time expires), and an oral exam (which is on-line, because you must answer the current question before proceeding to the next question).

Notice that we do not perform any operations comparing the relative values of elements, but merely require knowledge of their location. For this reason, we can assume that all the elements have been numbered sequentially from 1 to n and that the numbering can be determined easily by some hashing scheme. Thus, initially we have $S_i = \{i\}$ for i = 1 through n.

Our second observation is that the name of the set returned by *find* is actually fairly abitrary. All that really matters is that *find*(x) = *find*() if and only if x and are in the same set.

These operations are important in many graph theory problems and also in compilers which process equivalence (or type) declarations. We will see an application later.

There are two strategies to solve this problem. One ensures that the *find* instruction can be executed in constant worst-case time, and the other ensures that the *union* instruction can be executed in constant worst-case time. It has recently been shown that both cannot be done simultaneously in constant worst-case time.

We will now briefly discuss the first approach. For the *find* operation to be fast, we could maintain, in an array, the name of the equivalence class for each element. Then *find* is just a simple $O(1)$ lookup. Suppose we want to perform *union* (a, b). Suppose that a is in equivalence class i and b is in equivalence class j. Then we scan down the array, changing all is to j. Unfortunately, this scan takes (n). Thus, a sequence of n - 1 unions (the maximum, since then everything is in one set), would take (n^2) time. If there are (n^2) *find* operations, this performance is fine, since the total running time would then amount to $O(1)$ for each *union* or *find* operation over the course of the algorithm. If there are fewer *find*s, this bound is not acceptable.

One idea is to keep all the elements that are in the same equivalence class in a linked list. This saves time when updating, because we do not have to search through the entire array. This by itself does not reduce the asymptotic running time, because it is still possible to perform (n^2) equivalence class updates over the course of the algorithm.

If we also keep track of the size of each equivalence class, and when performing *unions* we change the name of the smaller equivalence class to the larger, then the total time spent for $n - 1$ merges is $O(n \log n)$. The reason for this is that each element can have its equivalence class changed at most $\log n$ times, since every time its class is changed, its new equivalence class is at least twice as large as its old. Using this strategy, any sequence of m *find*s and up to $n - 1$ *unions* takes at most $O(m + n \log n)$ time.

In the remainder of this chapter, we will examine a solution to the *union/find* problem that makes *unions* easy but *find*s hard. Even so, the running time for any sequences of at most m *find*s and up to $n - 1$ *unions* will be only a little more than $O(m + n)$.

8.3. Basic Data Structure

Recall that the problem does not require that a *find* operation return any specific name, just that *find*s on two elements return the same answer if and only if they are in the same set. One idea might be to use a tree to represent each set, since each element in a tree has the same root. Thus, the root can be used to name the set. We will represent each set by a tree. (Recall that a collection of trees is known as a *forest*.) Initially, each set contains one element. The trees we will use are not necessarily binary trees, but their representation is easy, because the only information we will need is a parent pointer. The name of a set is given by the node at the root. Since only the name of the parent is required, we can assume that this tree is stored implicitly in an array: each entry $p[i]$ in the array represents the parent of element i. If i is a root, then $p[i] = 0$. In the forest in Figure 8.1, $p[i] = 0$ for 1 i 8. As with heaps, we will draw the trees explicitly, with the understanding that an array is being used. Figure 8.1 shows the explicit representation. We will draw the root's parent pointer vertically for convenience.

To perform a *union* of two sets, we merge the two trees by making the root of one tree point to the root of the other. It should be clear that this operation takes constant time. Figures 8.2, 8.3, and 8.4 represent the forest after each of *union* (5,6) *union*(7,8), *union*(5,7), where we have adopted the convention that the new root after the *union*(x,y) is x. The implicit representation of the last forest is shown in Figure 8.5.

A *find*(x) on element x is performed by returning the root of the tree containing x. The time to perform this operation is proportional to the depth of the node representing x, assuming, of course, that we can find the node representing x in constant time. Using the strategy above, it is possible to create a tree of depth $n - 1$, so the worst-case running time of a *find* is $O(n)$. Typically, the running time is computed for a *sequence* of m intermixed instructions. In this case, m consecutive operations could take $O(mn)$ time in the worst case.

The code in Figures 8.6 through 8.9 represents an implementation of the basic algorithm, assuming that error checks have already been performed. In our routine, *unions* are performed on the roots of the trees. Sometimes the operation is performed by passing any two elements, and having the *union* perform two *find*s to determine the roots.

The average-case analysis is quite hard to do. The least of the problems is that the answer depends on how to define *average* (with respect to the *union* operation). For instance, in the forest in Figure 8.4, we could say that since there are five trees, there are 5 4 = 20 equally likely results of the next *union* (as any two different trees can be *union*ed). Of course, the implication of this model is that there is only a $\frac{2}{5}$ chance that the next *union* will involve the large tree. Another model might say that all *unions* between any two *elements* in different trees are equally likely, so a larger tree is more likely to be involved in the next *union* than a smaller tree. In the example above, there is an $\frac{8}{11}$ chance that the large tree is involved in the next union, since (ignoring symmetries) there are 6 ways in which to merge two elements in {1, 2, 3, 4}, and 16 ways to merge an element in {5, 6, 7, 8} with an element in {1, 2, 3, 4}. There are still more models and no general agreement on which is the best. The average running time depends on the model; (m), $(m \log n)$, and (mn) bounds have actually been shown for three different models, although the latter bound is thought to be more realistic.

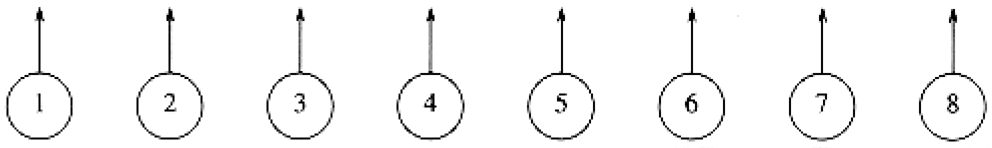


Figure 8.1 Eight elements, initially in different sets

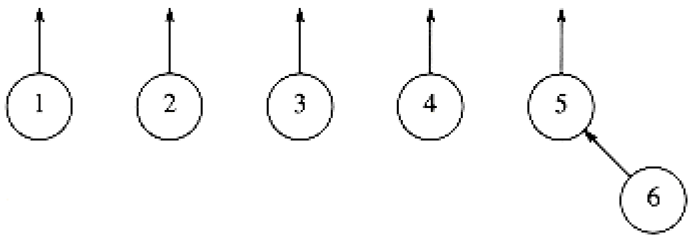


Figure 8.2 After union (5, 6)

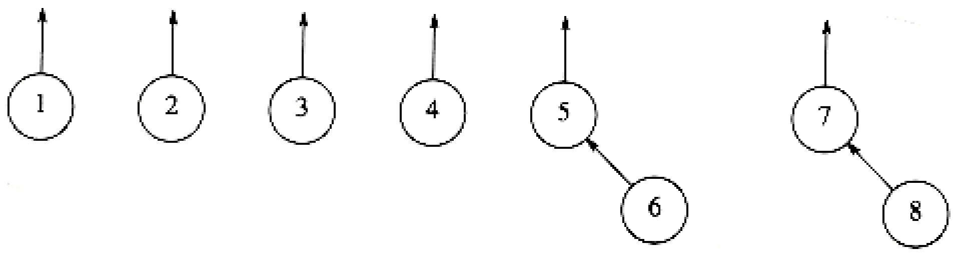


Figure 8.3 After union (7, 8)

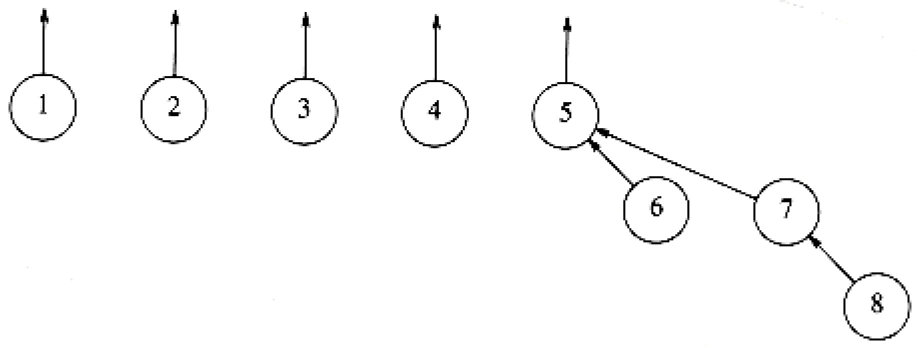
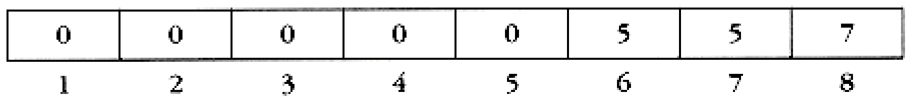


Figure 8.4 After union (5, 7)



0	0	0	0	0	5	5	7
1	2	3	4	5	6	7	8

Figure 8.5 Implicit representation of previous tree

```
typedef int DISJ_SET[ NUM_SETS+1 ];

typedef unsigned int set_type;

typedef unsigned int element_type;
```

Figure 8.6 Disjoint set type declaration

```
void

initialize( DISJ_SET S )

{

int i;

for( i = NUN_SETS; i > 0; i-- )

S[i] = 0;

}
```

Figure 8.7 Disjoint set initialization routine

```
/* Assumes root1 and root2 are roots. */

/* union is a C keyword, so this routine is named set_union. */

void

set_union( DISJ_SET S, set_type root1, set_type root2 )

{

S[root2] = root1;
}
```

Figure 8.8 Union (not the best way)

```
set_type

find( element_type x, DISJ_SET S )

{

if( S[x] <= 0 )

return x;

else

return( find( S[x], S ) );

}
```

Figure 8.9 A simple disjoint set find algorithm

Quadratic running time for a sequence of operations is generally unacceptable. Fortunately, there are several ways of easily ensuring that this running time does not occur.

8.4. Smart Union Algorithms

The *unions* above were performed rather arbitrarily, by making the second tree a subtree of the first. A simple improvement is always to make the smaller tree a subtree of the larger, breaking ties by any method; we call this approach *union-by-size*. The three *unions* in the preceding example were all ties, and so we can consider that they were performed by size. If the next operation were *union* (4, 5), then the forest in Figure 8.10 would form. Had the size heuristic not been used, a deeper forest would have been formed (Fig. 8.11).

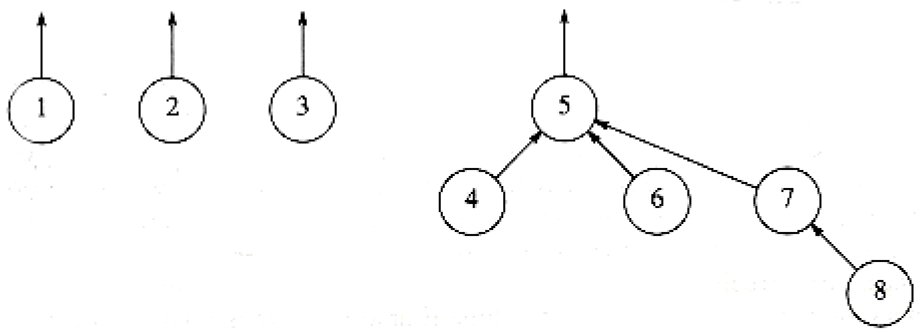


Figure 8.10 Result of union-by-size

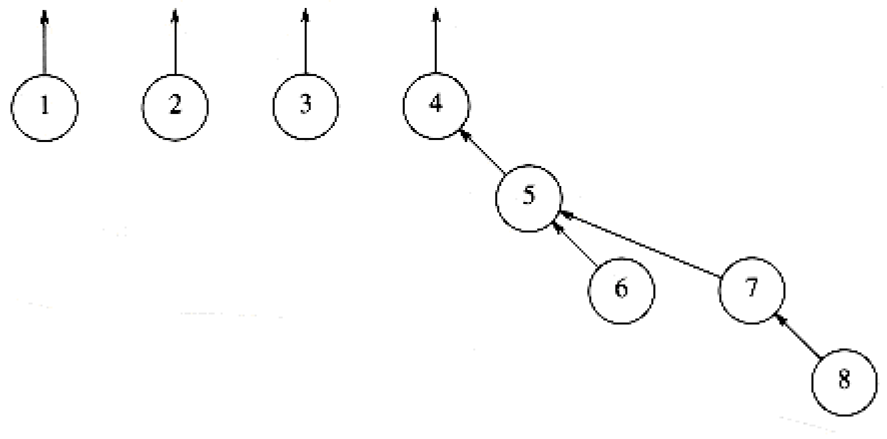


Figure 8.11 Result of an arbitrary union

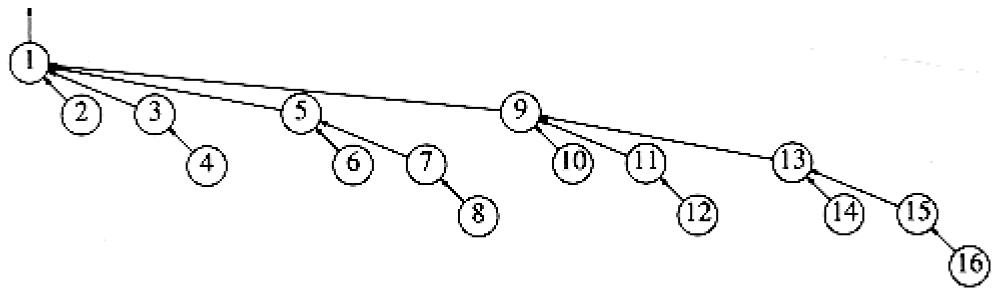


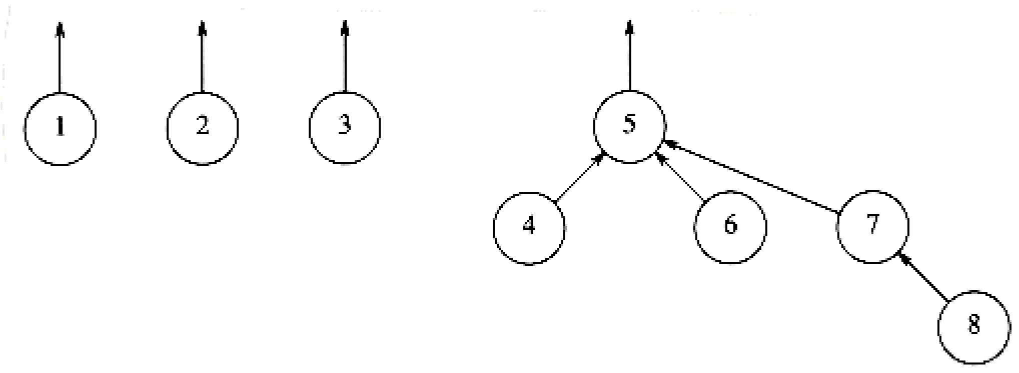
Figure 8.12 Worst-case tree for n = 16

We can prove that if *unions* are done by size, the depth of any node is never more than log *n*. To see this, note that a node is initially at depth 0. When its depth increases as a result of a *union*, it is placed in a tree that is at least twice as large as before. Thus, its depth can be increased at most log *n* times. (We used this argument in the quick-find algorithm at the end of Section 8.2.) This implies that the running time for a *find* operation is *O*(log *n*), and a sequence of *m* operations takes *O*(*m* log *n*). The tree in Figure 8.12 shows the worst tree possible after 16 *unions* and is obtained if all *unions* are between equal-sized trees (the worst-case trees are binomial trees, discussed in Chapter 6).

To implement this strategy, we need to keep track of the size of each tree. Since we are really just using an array, we can have the array entry of each root contain the *negative* of the size of its tree. Thus, initially the array representation of the tree is all -1s (and Fig 8.7 needs to be changed accordingly). When a *union* is performed, check the sizes; the new size is the sum of the old. Thus, union-by-size is not at all difficult to implement and requires no extra space. It is also fast, on average. For virtually all reasonable models, it has been shown that a sequence of *m* operations requires *O*(*m*) average time if union-by-size is used. This is because when random *unions* are performed, generally very small (usually one-element) sets are merged with large sets throughout the algorithm.

An alternative implementation, which also guarantees that all the trees will have depth at most *O*(log *n*), is *union-by-height*. We keep track of the height, instead of the size, of each tree and perform *unions* by making the shallow tree a subtree of the deeper tree. This is an easy algorithm, since the height of a tree increases only when two equally deep trees are joined (and then the height goes up by one). Thus, union-by-height is a trivial modification of union-by-size.

The following figures show a tree and its implicit representation for both union-by-size and union-by-height. The code in Figure 8.13 implements union-by-height.



-1	-1	-1	5	-5	5	5	7
1	2	3	4	5	6	7	8

0	0	0	5	-2	5	5	7
1	2	3	4	5	6	7	8

8.5. Path Compression

The *union/find* algorithm, as described so far, is quite acceptable for most cases. It is very simple and linear on average for a sequence of *m* instructions (under all models). However, the worst case of $O(m \log n)$ can occur fairly easily and naturally.

```
/* assume root1 and root2 are roots */

/* union is a C keyword, so this routine is named set_union */

void

set_union (DISJ_SET S, set_type root1, set_type root2 )

{

if( S[root2] < S[root1] )  /* root2 is deeper set */

S[root1] = root2;     /* make root2 new root */

else

{

if( S[root2] == S[root1] ) /* same height, so update */
S[root1]--;

S[root2] = root1; /* make root1 new root */

}

}
```

Figure 8.13 Code for union-by-height (rank)

For instance, if we put all the sets on a queue and repeatedly dequeue the first two sets and enqueue the union, the worst case occurs. If there are many more *finds* than *unions*, this running time is worse than that of the quick-find algorithm. Moreover, it should be clear that there are probably no more improvements possible for the *union* algorithm. This is based on the observation that any method to perform the unions will yield the same worst-case trees, since it must break ties arbitrarily. Therefore, the only way to speed the algorithm up, without reworking the data structure entirely, is to do something clever on the *find* operation.

The clever operation is known as *path compression*. Path compression is performed during a *find* operation and is independent of the strategy used to perform *unions*. Suppose the operation is *find*(*x*). Then the effect of path compression is that *every* node on the path from *x* to the root has its parent changed to the root. Figure 8.14 shows the effect of path compression after *find* (15) on the generic worst tree of Figure 8.12.

The effect of path compression is that with an extra two pointer moves, nodes 13 and 14 are now one position closer to the root and nodes 15 and 16 are now two positions closer. Thus, the fast future accesses on these nodes will pay (we hope) for the extra work to do the path compression.

As the code in Figure 8.15 shows, path compression is a trivial change to the basic *find* algorithm. The only change to the *find* routine is that *S*[*x*] is made equal to the value returned by *find*; thus after the root of the set is found recursively, *x* is made to point directly to it. This occurs recursively to every node on the path to the root, so this implements path compression. As we stated when we implemented stacks and queues, modifying a parameter to a function called is not necessarily in line with current software engineering rules. Some languages will not allow this, so this code may well need changes.

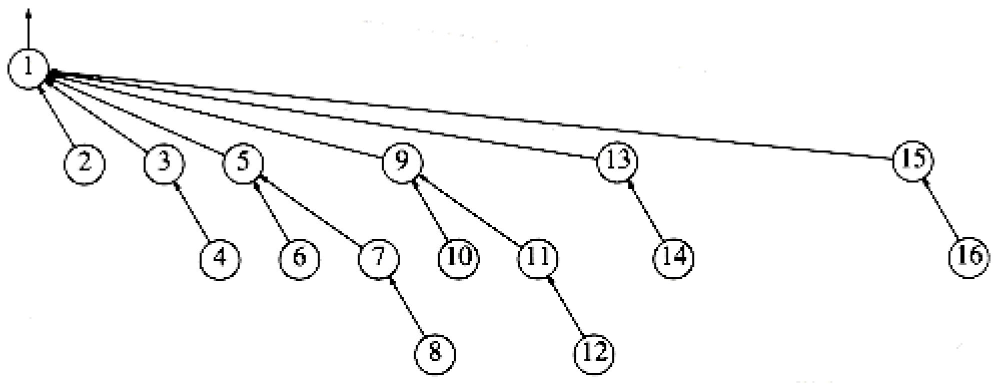


Figure 8.14 An example of path compression

```
set_type

find( element_type x, DISJ_SET S )

{

if( S[x] <= 0 )

return x;

else

return( S[x] = find( S[x], S ) );

}
```

Figure 8.15 Code for disjoint set find with path compression

When *unions* are done arbitrarily, path compression is a good idea, because there is an abundance of deep nodes and these are brought near the root by path compression. It has been proven that when path compression is done in this case, a sequence of *m* operations requires at most *O*(*m* log *n*) time. It is still an open problem to determine what the average-case behavior is in this situation.

Path compression is perfectly compatible with union-by-size, and thus both routines can be implemented at the same time. Since doing union-by-size by itself is expected to execute a sequence of *m* operations in linear time, it is not clear that the extra pass involved in path compression is worthwhile on average. Indeed, this problem is still open. However, as we shall see later, the combination of path compression and a smart union rule guarantees a very efficient algorithm in all cases.

Path compression is not entirely compatible with union-by-height, because path compression can change the heights of the trees. It is not at all clear how to re-compute them efficiently. The answer is do not!! Then the heights stored for each tree become estimated heights (sometimes known as *ranks*), but it turns out that union-by-rank (which is what this has now become) is just as efficient in theory as union-by-size. Furthermore, heights are updated less often than sizes. As with union-by-size, it is not clear whether path compression is worthwhile on average. What we will show in the next section is that with either union heuristic, path compression significantly reduces the worst-case running time.

8.6. Worst Case for Union-by-Rank and Path Compression

When both heuristics are used, the algorithm is almost linear in the worst case. Specifically, the time required in the worst case is (*m* (*m*, *n*)) (provided *m* *n*), where (*m*, *n*) is a functional inverse of Ackerman's function, which is defined below:*

$A(1, j) = 2^j$ for j 1

$A(i, 1) = A(i - 1, 2)$ for i 2

$A(i, j) = A(i - 1, A(i, j - 1))$ for i, j 2

*Ackerman's function is frequently defined with $A(1, j) = j + 1$ for j 1. the form in this text grows faster; thus, the inverse grows more slowly.

From this, we define

$(m, n) = \min\{i$ $1 | A(i,$ m/n $) > \log n\}$

You may want to compute some values, but for all practical purposes, (m, n) 4, which is all that is really important here. The single-variable inverse Ackerman function, sometimes written as log*n, is the number of times the logarithm of n needs to be applied until n 1. Thus, log* 65536 = 4, because log log log log 65536 = 1. log* 2^{65536} = 5, but keep in mind that 2^{65536} is a 20,000-digit number. (m, n) actually grows even slower then log* n. However, (m, n) is not a constant, so the running time is not linear.

In the remainder of this section, we will prove a slightly weaker result. We will show that any sequence of $m =$ (n) *union/find* operations takes a total of $O(m$ log* $n)$ running time. The same bound holds if union-by-rank is replaced with union-by-size. This analysis is probably the most complex in the book and one of the first truly complex worst-case analyses ever performed for an algorithm that is essentially trivial to implement.

8.6.1 Analysis of the Union/Find Algorithm

8.6.1 Analysis of the Union/Find Algorithm

In this section we establish a fairly tight bound on the running time of a sequence of $m =$ (n) *union/find* operations. The *unions* and *finds* may occur in any order, but *unions* are done by rank and *finds* are done with path compression.

We begin by establishing some lemmas concerning the number of nodes of *rank r*. Intuitively, because of the union-by-rank rule, there are many more nodes of small rank than large rank. In particular, there can be at most one node of rank log *n*. What we would like to do is to produce as precise a bound as possible on the number of nodes of any particular rank *r*. Since ranks only change when *unions* are performed (and then only when the two trees have the same rank), we can prove this bound by ignoring the path compression.

LEMMA 8.1.

When executing a sequence of union instructions, a node of rank r must have 2^r descendants (including itself).

PROOF:

By induction. the basis, $r = 0$, is clearly true. Let *T* be the tree of rank *r* with the fewest number of descendants and let *x* be *T*'s root. Suppose the last *union x* was involved in was between T_1 and T_2. Suppose T_1's root was *x*. If T_1 had rank *r*, then T_1 would be a tree of height *r* with fewer descendants than *T*, which contradicts the assumption that *T* is the tree with the smallest number of descendants. Hence the rank of T_1 $r - 1$. The rank of T_2 rank of T_1. Since *T* has rank *r* and the rank could only increase because of T_2, it follows that the rank of $T_2 = r - 1$. Then the rank of $T_1 = r - 1$. By the induction hypothesis, each tree has at least 2^{r-1} descendants, giving a total of 2^r and establishing the lemma.

Lemma 8.1 tells us that if no path compression is performed, then any node of rank *r* must have at least 2^r descendants. Path compression can change this, of course, since it can remove descendants from a node. However, when *unions* are performed, even with path compression, we are using the ranks, which are estimated heights. These ranks behave as though there is no path compression. Thus, when bounding the number of nodes of rank *r*, path compression can be ignored.

Thus, the next lemma is valid with or without path compression.

LEMMA 8.2.

The number of nodes of rank r is at most $n/2^r$.

PROOF:

Without path compression, each node of rank *r* is the root of a subtree of at least 2^r nodes. No node in the subtree can have rank *r*. Thus all subtrees of nodes of rank *r* are disjoint. Therefore, there are at most $n/2^r$ disjoint subtrees and hence $n/2^r$ nodes of rank *r*.

The next lemma seems somewhat obvious, but is crucial in the analysis.

LEMMA 8.3.

At any point in the union/find algorithm, the ranks of the nodes on a path from the left to a root increase monotonically.

PROOF:

The lemma is obvious if there is no path compression (see the example). If, after path compression, some node *v* is a descendant of *w*, then clearly *v* must have been a descendant of *w* when only *unions* were considered. Hence the rank of *v* is less than the rank of *w*.

Let us summarize the preliminary results. Lemma 8.2 tells us how many nodes can be assigned rank *r*. Because ranks are assigned only by *unions*, which have no idea of path compression, Lemma 8.2 is valid at any stage of the *union/find* algorithm--even in the midst of path compression. Figure 8.16 shows that while there are many nodes of ranks 0 and 1, there are fewer nodes of rank *r* as *r* gets larger.

Lemma 8.2 is tight, in the sense that it is possible for there to be $n/2^r$ nodes for any rank *r*. It is slightly loose, because it is not possible for the bound to hold for all ranks *r* simultaneously. While Lemma 8.2 describes the *number* of nodes in a rank *r*, Lemma 8.3 tells us their *distribution*. As one would expect, the rank of nodes is strictly increasing along the path from a leaf to the root.

We are now ready to prove the main theorem. Our basic idea is as follows: A *find* on any node *v* costs time proportional to the number of nodes on the path from *v* to the root. Let us, then, charge one unit of cost for every node on the path from *v* to the root for each *find*. To help us count the charges, we will deposit an imaginary penny into each node on the path. This is strictly an accounting gimmick, which is not part of the program. When the algorithm is over, we collect all the coins that have been deposited; this is the total cost.

As a further accounting gimmick, we deposit both American and Canadian pennies. We will show that during the execution of the algorithm, we can deposit only a certain number of American pennies during each *find*. We will also show that we can deposit only a certain number of Canadian pennies to each node. Adding these two totals gives us a bound on the total number of pennies that can be deposited.

We now sketch our accounting scheme in a little more detail. We will divide the nodes by their ranks. We then divide the ranks into rank groups. On each *find*, we will deposit some American coins into the general kitty and some Canadian coins into specific vertices. To compute the total number of Canadian coins deposited, we will compute the deposits per node. By adding up all the deposits for each node in rank *r*, we will get the total deposits per rank *r*. Then we will add up all the deposits for each rank *r* in group *g* and thereby obtain the total deposits for each rank group *g*. Finally, we add up all the deposits for each rank group *g* to obtain the total number of Canadian coins deposited in the forest. Adding this to the number of American coins in the kitty gives us the answer.

We will partition ranks into groups. Ranks *r* goes into group $G(r)$, and *G* will be determined later. The largest rank in any rank group *g* is $F(g)$, where $F = G^{-1}$ is the *inverse* of *G*. The number of ranks in any rank group, $g > 0$, is thus $F(g) - F(g - 1)$. Clearly $G(n)$ is a very loose upper bound on the largest rank group. As an example, suppose that we partitioned the ranks as in Figure 8.7. In this case, $G(r) = \lceil \sqrt{r} \rceil$. The largest rank in group *g* is $F(g) = g^2$, and observe that group $g > 0$ contains ranks $F(g - 1) + 1$ through $F(g)$ inclusive. This formula does not apply for rank group 0, so for convenience we will ensure that rank group 0 contains only elements of rank 0. Notice that the groups are made of consecutive ranks.

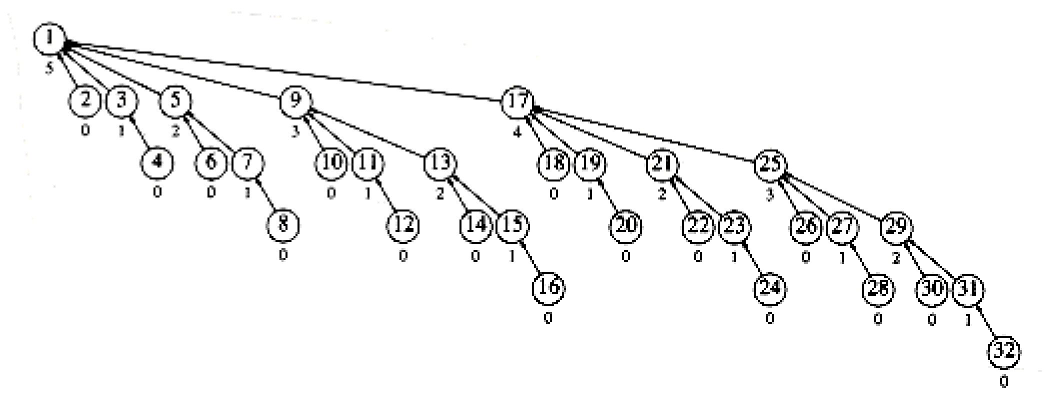


Figure 8.16 A large disjoint set tree (numbers below nodes are ranks)

As mentioned before, each *union* instruction takes constant time, as long as each root keeps track of how big its subtrees are. Thus, *unions* are essentially free, as far as this proof goes.

Each *find*(*i)* takes time proportional to the number of vertices on the path from the vertex representing *i* to the root. We will thus deposit one penny for each vertex on the path. If this is all we do, however, we cannot expect much of a bound, because we are not taking advantage of path compression. Thus, we need to take advantage of path compression in our analysis. We will use fancy accounting.

For each vertex, *v*, on the path from the vertex representing *i* to the root, we deposit one penny under one of two accounts:

1. If *v* is the root, or if the parent of *v* is the root, or if the parent of *v* is in a different rank group from *v*, then charge one unit under this rule. This deposits an American penny into the kitty.

2. Otherwise deposit a Canadian penny into the vertex.

LEMMA 8.4.

For any find (v), the total number of pennies deposited, either into the kitty or into a vertex, is exactly equal to the number of nodes on the path from v to the root.

PROOF:

Obvious.

Thus all we need to do is to sum all the American pennies deposited under rule 1 with all the Canadian pennies deposited under rule 2.

We are doing at most *m finds*. We need to bound the number of pennies that can be deposited into the kitty during a *find.*

LEMMA 8.5

Over the entire algorithm, the total deposits of American pennies under rule 1 amount to m(G(n) + 2).

```
Group         Rank
------------------------------
```

0	0
1	1
2	2,3,4
3	5 through 9
4	10 through 16
i	$(i-1)^2 + 1$ through i^2

Figure 8.17 Possible partitioning of ranks into groups

PROOF:

This is easy. For any *find*, two American pennies are deposited, because of the root and its child. By Lemma 8.3, the vertices going up the path are monotonically increasing in rank, and since there are at most $G(n)$ rank groups, only $G(n)$ other vertices on the path can qualify as a rule 1 deposit for any particular *find*. Thus, during any one find, at most $G(n) + 2$ American pennies can be placed in the kitty. Thus, at most $m(G(n) + 2)$ American pennies can be deposited under rule 1 for a sequence of *m finds*.

To get a good estimate for all the Canadian deposits under rule 2, we will add up the deposits by vertices instead of by *find* instructions. If a coin is deposited into vertex v under rule 2, v will be moved by path compression and get a new parent of higher rank than its old parent. (This is where we are using the fact that path compression is being done.) Thus, a vertex v in rank group $g > 0$ can be moved at most $F(g) - F(g - 1)$ times before its parent gets pushed out of rank group g, since that is the size of the rank group.* After this happens, all future charges to v will go under rule 1.

*This can be reduced by 1. We do not for the sake of clarity; the bound is not improved by being more careful here.

LEMMA 8.6.

The number of vertices, $N(g)$, in rank group $g > 0$ is at most $n/2^{F(g - 1)}$.

PROOF:

By Lemma 8.2, there are at most $n/2^r$ vertices of rank r. Summing over the ranks in group g, we obtain

$$
\begin{aligned}
N(g) &\le \sum_{r=F(g-1)+1}^{F(g)} \frac{n}{2^r} \\
&\le \sum_{r=F(g-1)+1}^{\infty} \frac{n}{2^r} \\
&\le n \sum_{r=F(g-1)+1}^{\infty} \frac{1}{2^r} \\
&\le \frac{n}{2^{F(g-1)+1}} \sum_{s=0}^{\infty} \frac{1}{2^s} \\
&\le \frac{2n}{2^{F(g-1)+1}} \\
&\le \frac{n}{2^{F(g-1)}}
\end{aligned}
$$

LEMMA 8.7.

The number of Canadian pennies deposited into all vertices in rank group g is at most $nF(g)/2^{F(g - 1)}$.

PROOF:

Each vertex in the rank group can receive at most $F(g) - F(g - 1)$ $F(g)$ Canadian pennies while its parent stays in its rank group, and Lemma 8.6 tells how many such vertices there are.

The result is obtained by a simple multiplication.

LEMMA 8.8.

The total deposit under rule 2 is at most n $\sum_{g=1}^{G(n)} F(g)/2^{F(g-1)}$ *Canadian pennies.*

PROOF:

Because rank group 0 contains only elements of rank 0, it cannot contribute to rule 2 charges (it cannot have a parent in the same rank group). The bound is obtained by summing the other rank groups.

Thus we have the deposits under rules 1 and 2. The total is

$$m(G(n) + 2) + n \sum_{g=1}^{G(n)} F(g)/2^{F(g-1)}$$

(8.1)

We still have not specified $G(n)$ or its inverse $F(n)$. Obviously, we are free to choose virtually

anything we want, but it makes sense to choose $G(n)$ to minimize the bound above. However, if $G(n)$ is too small, then $F(n)$ will be large, hurting the bound. An apparently good choice is to choose $F(i)$ to be the function recursively defined by $F(0) = 0$ and $F(i) = 2^{F(i - 1)}$. This gives $G(n) = 1 +[\log^* n]$. Figure 8.18 shows how this partitions the ranks. Notice that group 0 contains only rank 0, which we required in the previous lemma. F is very similar to the single-variable Ackerman function, which differs only in the definition of the base case ($F(0) = 1$).

THEOREM 8.1.

The running time of m unions and finds is O(m log n).*

Group	Rank
0	0
1	1
2	2
3	3,4
4	5 through 16
5	17 through 2^{16}
6	65537 through 2^{65536}
7	truly huge ranks

Figure 8.18 Actual partitioning of ranks into groups used in the proof

PROOF:

Plug in the definitions of F and G into Equation (8.1). The total number of American pennies is $O(mG(n)) = O(m \log^* n)$. The total number of Canadian pennies is

$n \sum_{g=1}^{G(n)} F(g)/2^{F(g-1)} = n \sum_{g=1}^{G(n)} 1 = nG(n) = O(n \log^* n)$. Since $m =$ (n), the bound follows.

What the analysis shows is that there are few nodes that could be moved frequently by path compression, and thus the total time spent is relatively small.

8.7. An Application

As an example of how this data structure might be used, consider the following problem. We have a network of computers and a list of bidirectional connections; each of these connections allows a file transfer from one computer to another. Is it possible to send a file from any computer on the network to any other? An extra restriction is that the problem must be solved *on-line*. Thus, the list of connections is presented one at a time, and the algorithm must be prepared to give an answer at any point.

An algorithm to solve this problem can initially put every computer in its own set. Our invariant is that two computers can transfer files if and only if they are in the same set. We can see that the ability to transfer files forms an equivalence relation. We then read connections one at a time. When we read some connection, say (u, v), we test to see whether u and v are in the same set and do nothing if they are. If they are in different sets, we merge their sets. At the end of the algorithm, the graph is connected if and only if there is exactly one set. If there are m connections and n computers, the space requirement is $O(n)$. Using union-by-size and path compression, we obtain a worst-case running time of $O(m\,(m, n))$, since there are *2m finds* and at most $n - 1$ *unions*. This running time is linear for all practical purposes.

We will see a much better application in the next chapter.

Summary

We have seen a very simple data structure to maintain disjoint sets. When the *union* operation is performed, it does not matter, as far as correctness is concerned, which set retains its name. A valuable lesson that should be learned here is that it can be very important to consider the alternatives when a particular step is not totally specified. The *union* step is flexible; by taking advantage of this, we are able to get a much more efficient algorithm.

Path compression is one of the earliest forms of *self-adjustment*, which we have seen elsewhere (splay trees, skew heaps). Its use is extremely interesting, especially from a theoretical point of view, because it was one of the first examples of a simple algorithm with a not-so-simple worst-case analysis.

Exercises

8.1 Show the result of the following sequence of instructions: *union*(1, 2), *union*(3, 4), *union*(3, 5), *union*(1, 7), *union*(3, 6), *union*(8, 9), *union*(1, 8), *union*(3, 10), *union*(3, 11), *union*(3, 12), *union*(3, 13), *union*(14, 15), *union*(16, 17), *union*(14, 16), *union*(1, 3), *union*(1, 14), when the *unions* are

a. performed arbitrarily

b. performed by height

c. performed by size

8.2 For each of the trees in the previous exercise, perform a *find* with path compression on the deepest node.

8.3 Write a program to determine the effects of path compression and the various *unioning* strategies. Your program should process a long sequence of equivalence operations using all six of the possible strategies.

8.4 Show that if *unions* are performed by height, then the depth of any tree is $O(\log n)$.

8.5 a. Show that if $m = n^2$, then the running time of m *union/find* operations is $O(m)$.

b. Show that if $m = n \log n$, then the running time of m *union/find* operations is $O(m)$.

*c. Suppose $m =$ $(n \log \log n)$. What is the running time of m *union/find* operations?

*d. Suppose $m =$ $(n \log^* n)$. What is the running time of m *union/find* operations?

8.6 Show the operation of the program in Section 8.7 on the following graph: (1,2), (3,4), (3,6), (5,7), (4,6), (2,4), (8,9), (5,8). What are the connected components?

8.7 Write a program to implement the algorithm in Section 8.7.

*8.8 Suppose we want to add an extra operation, *deunion*, which undoes the last *union* operation that has not been already undone.

a. Show that if we do union-by-height and *finds* without path compression, then *deunion* is easy and a sequence of m *union, find*, and *deunion* operations take $O(m \log n)$ time.

b. Why does path compression make *deunion* hard?

**c. Show how to implement all three operations so that the sequence of m operations takes $O(m \log n/\log \log n)$ time.

* 8.9 Suppose we want to add an extra operation, *remove*(x), which removes x from its current set and places it in its own. Show how to modify the *union/find* algorithm so that the running time of a sequence of m *union, find*, and *remove* operations is $O(m$ $(m, n))$.

**8.10 Give an algorithm that takes as input an n-vertex tree and a list of n pairs of vertices and determines for each pair (v, w) the closest common ancestor of v and w. Your algorithm should run in $O(n \log^* n)$.

*8.11 Show that if all of the *unions* precede the *finds*, then the disjoint set algorithm with path compression requires linear time, even if the *unions* are done arbitrarily.

**8.12 Prove that if *unions* are done arbitrarily, but path compression is performed on the *finds*, then the worst-case running time is $(m \log n)$.

8.13 Prove that if *unions* are done by size and path compression is performed, the worst-case running time is $O(m \log* n)$.

8.14 Suppose we implement partial path compression on *find*(i) by making every other node on the path from i to the root point to its grandparent (where this makes sense). This is known as *path halving*.

a. Write a procedure to do this.

b. Prove that if path halving is performed on the *finds* and either union-by-height or union-by-size is used, the worst-case running time is $O(m \log* n)$.

References

Various solutions to the *union/find* problem can be found in [5], [8], and [10]. Hopcroft and Ullman showed the $O(m \log* n)$ bound of Section 8.6. Tarjan [14] obtained the bound $O(m\ (m, n))$. A more precise (but asymptotically identical) bound for $m < n$ appears in [2] and [17]. Various other strategies for path compression and *unions* also achieve the same bound; see [17] for details.

A lower bound showing that under certain restrictions $(m\ (m, n))$ time is required to process m *union/find* operations was given by Tarjan [15]. Identical bounds under less restrictive conditions have been recently shown in [6] and [13].

Applications of the *union/find* data structure appear in [1] and [9]. Certain special cases of the *union/find* problem can be solved in $O(m)$ time [7]. This reduces the running time of several algorithms, such as [1], graph dominance, and reducibility (see references in Chapter 9) by a factor of (m, n). Others, such as [9] and the graph connectivity problem in this chapter, are unaffected. The paper lists 10 examples. Tarjan has used path compression to obtain efficient algorithms for several graph problems [16].

Average-case results for the *union/find* problem appear in [4], [11], and [19]. Results bounding the running time of any single operation (as opposed to the entire sequence) appear in [3] and [12].

Exercise 8.8 is solved in [18].

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